INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 2, 263–276 (1982)

LONGSHORE BOUNDARY CONDITIONS FOR NUMERICAL WAVE MODELS

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SUMMARY

The problem of truncating nearshore finite element wave models is addressed. Incorrect treatment of the artificial boundaries of the model will cause spurious wave reflections. Three methods for dealing with these boundaries: application of constraints, use of the Smith condition and longshore dampers, are proposed. Numerical results show the dampers to be the best method.

KEY WORDS Finite Elements Waves Coastline Dampers

INTRODUCTION

We consider the problem in which we wish to predict numerically patterns of waves in a region adjacent to a long coastline.[1,2,4] The numerical model will thus have four boundaries, enclosing the region of interest: the coastline, two arbitrary boundaries roughly normal to the coast and a seaward boundary, as sketched in Figure 1. In this paper we are concerned with accurate modelling of the two artificial boundaries normal to the coast. The effect of the seaward boundary is usually less significant.

Waves from deep water are incident upon the coastline at some angle θ. As they encounter the shallower water they refract; so that, in general, the angle of incidence becomes smaller. At a certain point they begin to break and energy is changed into heat and sound in the surf zone.

In the region of interest the problem can be complicated by wave reflections leading to partial standing waves, diffraction and resonance effects. In the traditional method of ray tracing these effects can be modelled only with difficulty. In what follows we assume that the domain of interest is modelled using finite elements, which can cope effectively with these phenomena. Most of the discussion would apply equally to finite differences.

FINITE ELEMENT MODELLING OF WAVES

The starting point for the numerical model is the wave equation of Berkhoff[3]

$$\nabla(cc_g\nabla\phi)+\frac{\omega^2 c_g}{c}\phi=0 \tag{1}$$

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0271–2091/82/030263–14$01.40

Received 3 November 1981
Revised 24 November 1981

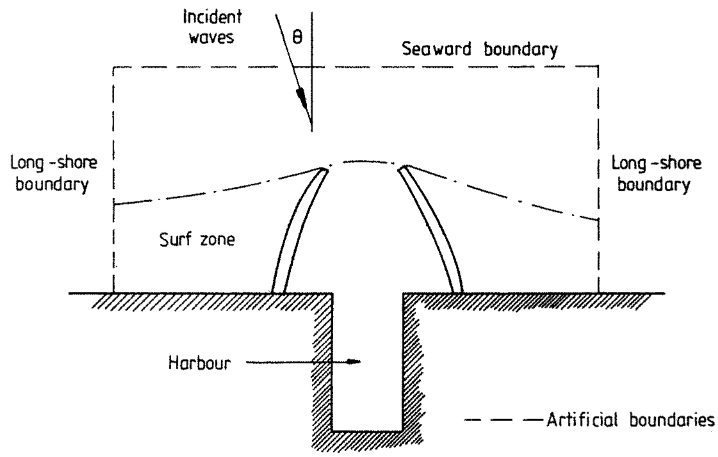

Figure 1. Geometry of coastline—showing boundaries of numerical model

which gives the shallow water wave equation as a special case. Here ϕ is a complex velocity potential, ω is the frequency, c is the wave celerity and $c=\omega/k$. The dispersion relation

$$\omega^2 = gk \tanh kh \tag{2}$$

gives the frequency ω in terms of acceleration due to gravity g, depth h and wavenumber k. Wavenumber k is defined as $2\pi/\lambda$, where λ is the surface wavelength. The group velocity c_g is given by

$$c_g = \frac{1}{2}\left(1+\frac{2kh}{\sinh 2kh}\right) \tag{3}$$

The surface elevation η can be obtained from the velocity potential ϕ by

$$\eta = \frac{i\omega}{g}\phi \tag{4}$$

Also the orbital velocities u and v in the waves can be obtained from the velocity potential ϕ.

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y} \tag{5}$$

All quantities are factored by exp ($i\omega t$), as the problem is completely periodic. The functional corresponding to equation (1) is

$$\Pi = \iint_{\Omega} \left[(\nabla\phi \, . \, cc_g \nabla\phi) + \frac{\omega^2 c_g}{c}\phi^2\right] d\Omega \tag{6}$$

Now this functional can be used as the basis for a finite element model. Diffraction and refraction effects are modelled automatically using such elements. If eight-noded elements are used then about four are needed per wavelength for reasonable accuracy. The natural boundary condition to the functional is

$$\frac{\partial \phi}{\partial n} = 0 \tag{7}$$

which implies zero velocity normal to the boundary, and complete wave reflection.

A finite element shape function $\mathbf{N}$ is now introduced such that within an element

$$\phi = \mathbf{N}\boldsymbol{\phi} \tag{8}$$

where $\boldsymbol{\phi}$ is the set of nodal values of the velocity potential. In the program used for this work eight-noded quadrilateral elements were adopted. For further details of the theory of finite elements see Zienkiewicz.[11] In the finite element model the known incident waves give rise to 'loading' terms.[5] The resulting element matrices are assembled and solved for the values of ϕ at all nodes. More details of the process of finite element modelling of waves are given in Reference 5. Because linear wave theory is being used all the other required quantities can be obtained from ϕ.

Where waves are to be absorbed, for example on open boundaries, or at permeable breakwaters or on beaches, a 'damper' boundary condition can be introduced. This boundary condition is

$$\frac{\partial \phi}{\partial n} + ik\phi = 0 \tag{9}$$

If a line integral is added to the functional (4), consisting of

$$\int_S \frac{ik\phi^2}{2}\,\mathrm{d}S$$

on these boundaries, S, on which waves are to be absorbed, then the natural boundary condition on S becomes equation (9). Partial absorption can be dealt with by multiplying the line integral by some factor between 0 and 1, these two values representing total reflection and total absorption.

When the water becomes shallow the waves break and their energy is dissipated. This is an extremely complicated effect and is difficult to model accurately. In our model the waves are simply absorbed at the beach by line dampers. The depths close to the shore are altered as shown in Figure 2 so that they remain finite. Otherwise the wavenumber $k = \omega/\sqrt{(gh)}$ tends to infinity and the wavelength tends to zero. Since it is necessary to use about four elements per wavelength this obviously causes numerical difficulties. Instead, if the depth is altered as described, the wavelength remains finite. The waves are propagated right up to the shoreline. Because they have refracted on the way up the beach they are, for all practical purposes, normally incident. These normal waves are then absorbed using dampers of the type described by equation (9). The k value is derived from the depth h at the shoreline.

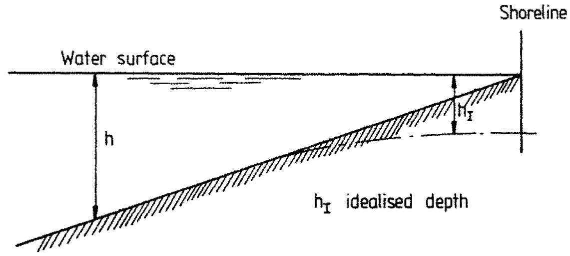

Figure 2. Cross-section of beach—showing idealized depths

REPEATABILITY BOUNDARY CONDITION

If in modelling a longshore problem it is known that the flow profile at one artificial boundary is the same as the flow profile at the second boundary, then it is possible to apply a repeatability condition. The idea has also been applied to the modelling of longshore currents by Zienkiewicz and Heinrich.[9] In this case the shallow water equations are solved with the repeatibility condition specifying that the velocities at corresponding nodal points on the artificial boundaries are identical. The condition is restrictive in that the beach slope and water depth need to be identical at each end of the model. If, in extending the length of the model, there are no changes in the boundary values of the velocities then a situation has been achieved which is equivalent to modelling an infinitely long beach.

Modelling the artificial boundaries of the wave problem using a repeatability technique is more difficult as it involves enforcing the phase shift which occurs with a non-normal angle of wave incidence. Although the amplitude of the wave is constant along a line parallel to the shoreline for a plane beach there is a phase shift related to the angle of wave incidence. From Figure 3 it can be seen that the wave potentials at two corresponding nodes are given by

$$\phi_1 = \phi_2 e^{i\alpha} \tag{10}$$

where α is the ratio of the wavelength to the longshore component of the wavelength. 1 and 2 are the ends of the model as shown in Figure 4. This is only true for a beach with constant slope. Snell's law of refraction is applicable. For water waves the law reduces to

$$\frac{\sin\theta}{c} = \text{constant} \tag{11}$$

Orris and Petyt[7] encountered a similar condition in the analysis of vibration and response in periodic structures. They considered the displacements on one side of one of a number of identically linked substructures was a constant ratio times the displacements on the other

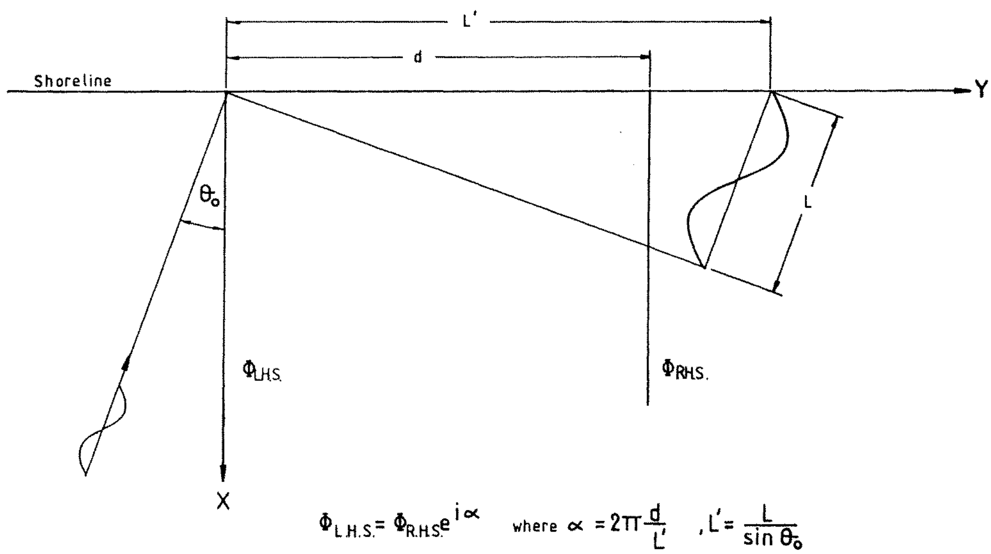

Figure 3. Phase shift along section of plane beach

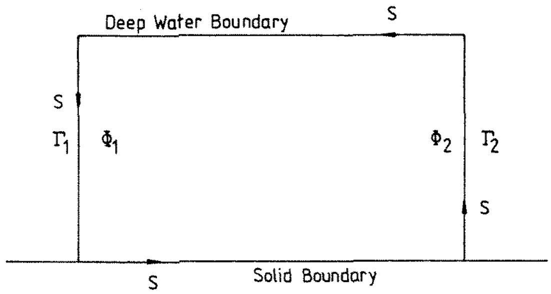

Figure 4. Boundary definition for variational calculus theory

side. That is, the displacements u on the left-hand side are linked to displacements on the right-hand side by

$$u_{\mathrm{RHS}} = u_{\mathrm{LHS}} \mathrm{e}^{\mu} \tag{12}$$

where μ is the complex propagation constant. From the repeatability considerations if the harbour, or breakwater, is not influencing the boundaries then it seems possible to consider the section of beach of interest as one of a series of identically linked sections.

To apply the boundary condition given by equation (10) a Lagrangian Multiplier technique was used. The additional boundary condition to be applied becomes

$$\lambda(\phi_1 - \phi_2 \mathrm{e}^{i\alpha}) = 0 \tag{13}$$

where λ is the Lagrangian Multiplier and physically represents the flux of energy at the boundary. Equation (13) is added to the wave functional by a term

$$\int_S \lambda(\phi_1 - \phi_2 \mathrm{e}^{i\alpha})\, \mathrm{d}S \tag{14}$$

The variation is now performed with respect to λ as well as to ϕ

$$\delta\Pi = \frac{\partial \Pi}{\partial \phi} \, . \, \delta\phi + \frac{\partial \Pi}{\partial \lambda} \, . \, \delta\lambda \tag{15}$$

the second term yielding equation (13) in the finite element approximation. A three-noded element was used at the boundary, the third nodal variable being the Lagrange Multiplier. The first two nodal variables were the two velocity potentials at corresponding nodes at the two ends of the model.

The condition when tested with $n\pi$ phase shifts across the model yielded excellent agreement with the analytical solution. This is demonstrated in Figure 5. However, as the phase shift approached $(n+1/2)\pi$ the solution progressively deteriorated. This deterioration is shown by means of a phase shift diagram, Figure 6. On re-examination of the theory it was evident that a further gradient boundary condition was only being applied correctly in the $n\pi$ phase change situations.

Consider the boundaries Γ_1 and Γ_2, equivalent to the artificial boundaries, shown in Figure 4. We are applying $\phi_1 = \phi_2 \mathrm{e}^{i\alpha}$ and hence the gradient boundary condition is

$$\frac{\partial \phi_1}{\partial n} = -\frac{\partial \phi_2}{\partial n} \mathrm{e}^{i\alpha} \tag{16}$$

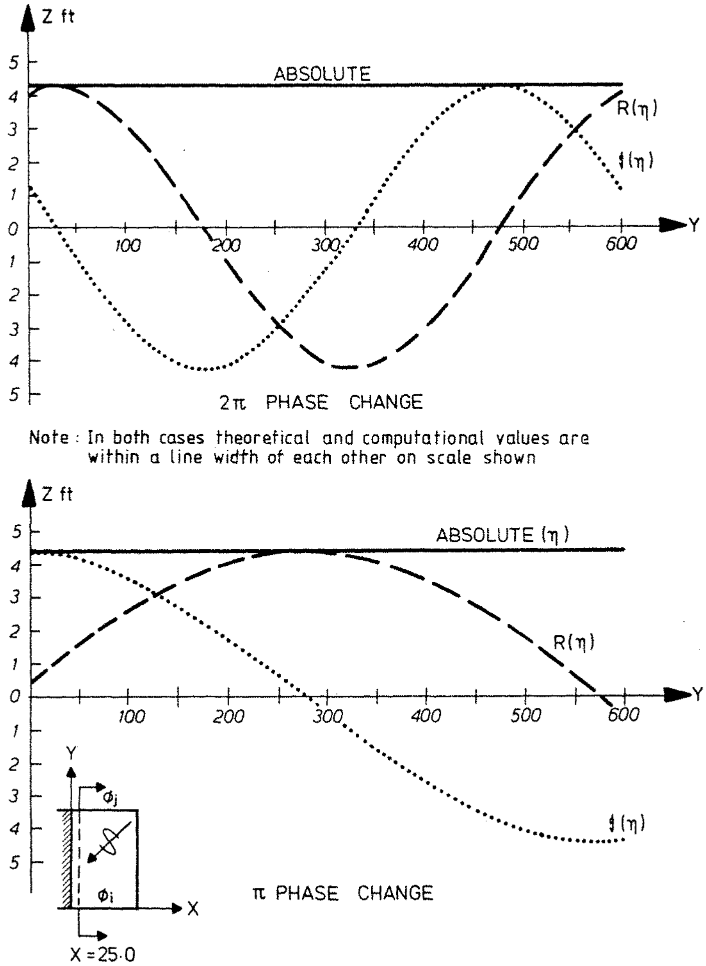

Figure 5. Wave elevation parallel to shoreline using imposed phase shift boundary conditions $\phi_i = \phi_j e^{i\gamma}$

where n is the outward normal. Taking the additional Lagrange Multiplier terms added to the boundary as

$$\int_{\Gamma_1} \lambda\phi_1 \, d\Gamma \quad \text{and} \quad \int_{\Gamma_2} -\lambda\phi_2 e^{i\alpha} \, d\Gamma \tag{17}$$

the variation of λ leads to equation (10). However, the wave functional variation with respect to ϕ_1 leads to

$$\frac{\partial\phi_1}{\partial n} + \lambda = 0 \tag{18}$$

and variation with respect to ϕ_2 leads to

$$\frac{\partial\phi_2}{\partial n} - \lambda e^{i\alpha} = 0 \tag{19}$$

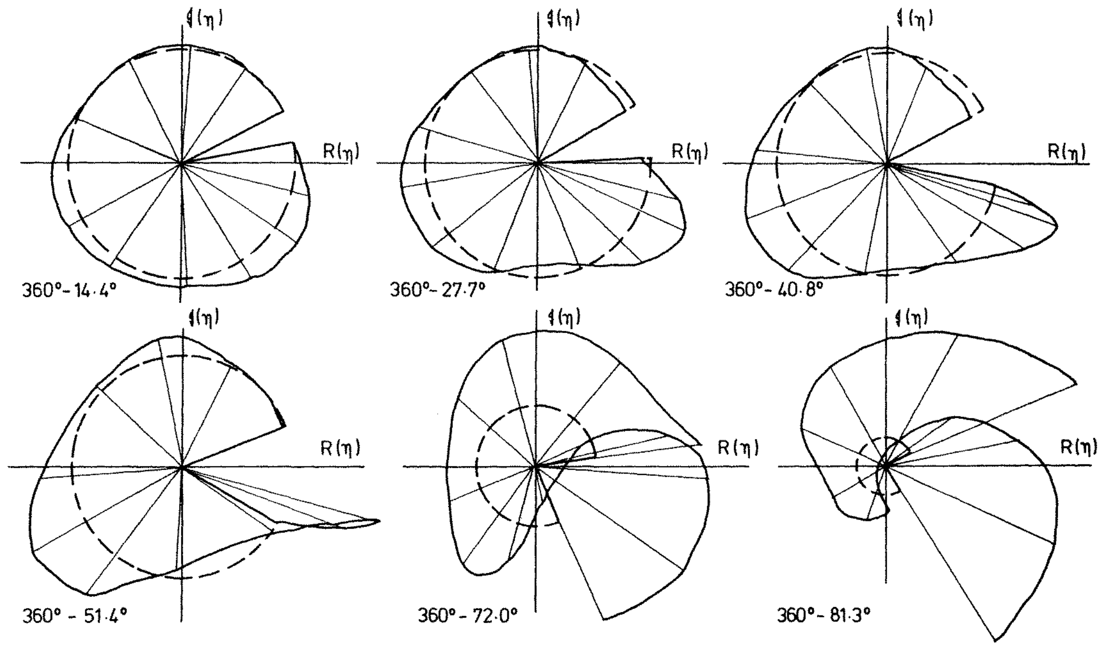

Figure 6. Vector diagrams of phase angle parallel to shoreline using imposed phase shift boundary conditions. Dashed circle is theoretical solution. Scale halves in fifth and sixth diagrams

Upon substituting (18) in (19) we obtain

$$\frac{\partial \phi_1}{\partial n} = -\frac{\partial \phi_2}{\partial n} e^{-i\alpha} \tag{20}$$

On comparing (20) with (16) it is evident that the application of Lagrange Multipliers is only valid when

$$e^{i\alpha} = e^{-i\alpha}$$

or

$$e^{2i\alpha} = 1$$

that is when a phase shift of $n\pi$ occurs across the model (without the application of a further gradient condition).

THE SMITH NON-REFLECTING BOUNDARY CONDITION

An alternative to application of the repeatability condition is the removal of spurious reflections caused at the artificial boundary. Such techniques are collectively known as non-reflecting boundary conditions.

Smith[8] describes a method of eliminating wave reflections from artificial boundaries by the addition of two solutions to the problem, one using the Neumann boundary condition and one using the Dirichlet condition.

The Neumann boundary condition

$$\frac{\partial \phi}{\partial n} = 0 \tag{21}$$

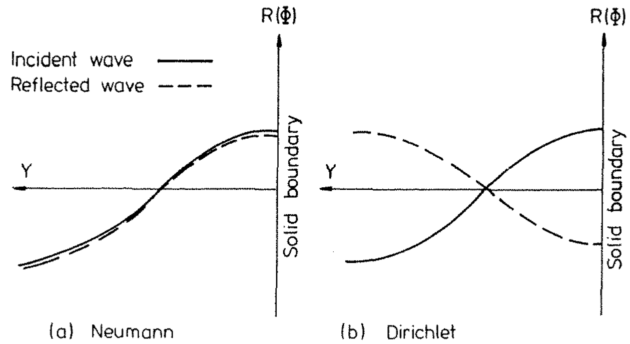

Figure 7. Reflected waves resulting from application of Neumann and Dirichlet conditions to a solid boundary

is the usual boundary condition applied by the wave program on solid boundaries. It imposes a continuity of slope and thus the reflected wave is a mirror of the incident wave. The Dirichlet condition

$$\phi = 0 \tag{22}$$

results in a reflected wave 180° out of phase with the incident wave. These are sketched in Figure 7. Consequently the addition of the two solutions should eliminate the reflected wave. The Dirichlet condition was applied to the finite element program by adding in a large term to the boundary node 'stiffness' matrix, thus enforcing the boundary velocity potential to be effectively zero. However, the addition of two solutions did not eliminate the wave. As two boundaries were in operation two additional runs using mixed boundary conditions were also made and added to the previous solutions without improvement. The results are shown in Figure 8.

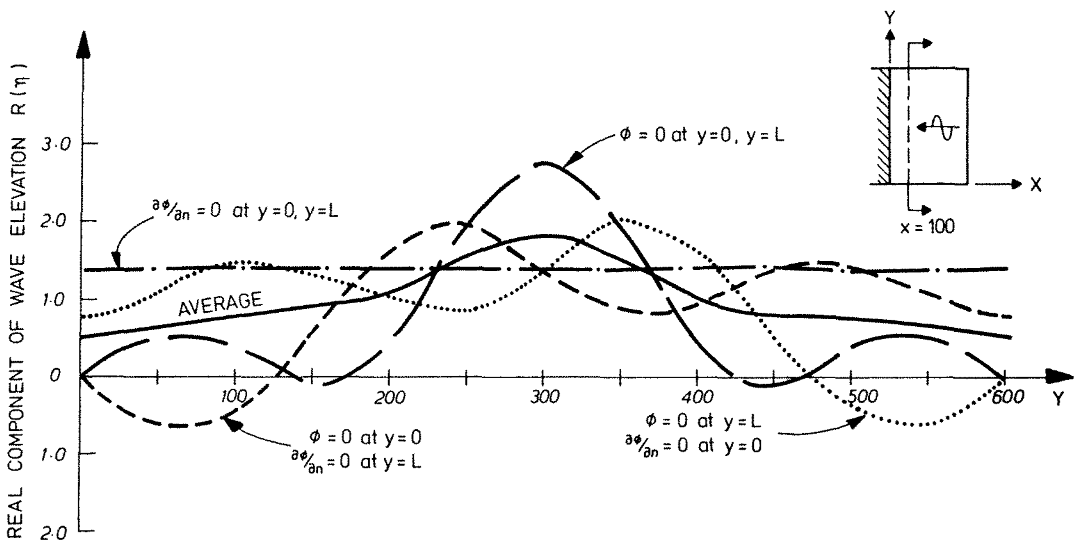

Figure 8. Smith boundary condition. Four solutions using boundary conditions as shown. Wave direction normal to shoreline, plots parallel to shoreline

The wavelength is constantly decreasing as the shoreline is approached and it is therefore possible to regard the problem as one of multiple reflections in which case as Smith states the reflections cannot be eliminated.

LONGSHORE DAMPERS

The reflected waves resulting from waves meeting an artificial boundary represent an excess of energy in the boundary region. It should be possible to remove this excess energy directly by damping out the component of the reflected wave which should have travelled on through the artificial boundary. By the same reasoning at the other boundary there is a lack of energy because the wave component that should be transmitted into the domain is absent. This wave component can be supplied by adding a negative damper which generates the incoming wave. The overall effect is that the energy removed from one boundary is added into the other.

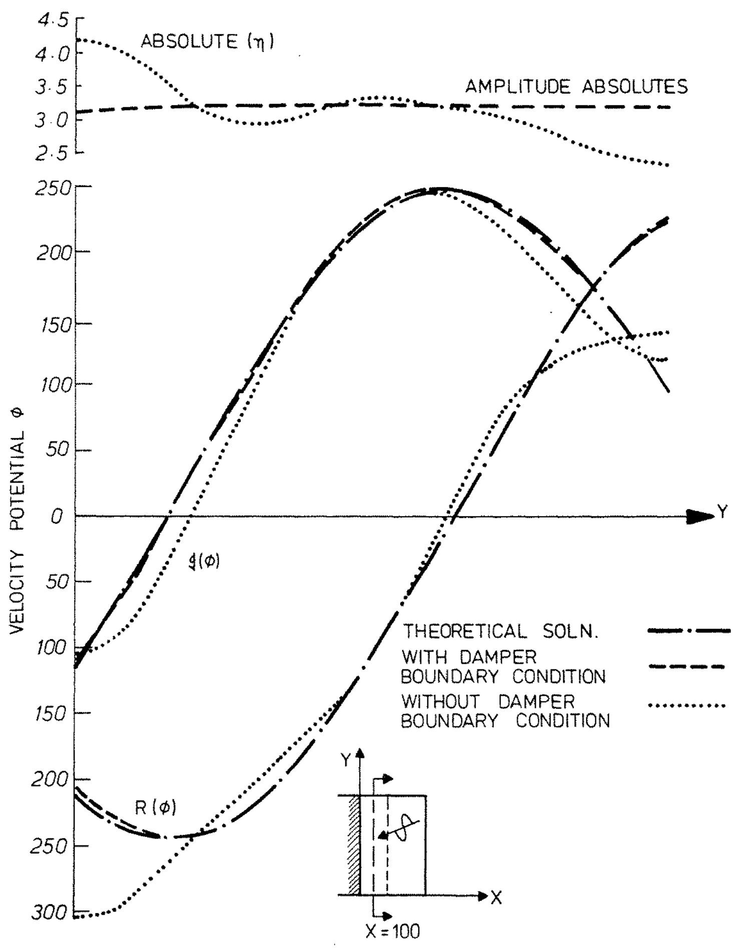

Figure 9. Wave potential parallel to shoreline with and without damping elements on artificial boundaries. 12° angle of wave incidence

It is well known that dampers can be used to absorb outgoing waves, having been introduced by Zienkiewicz and Newton[10] for pressure waves and Lysmer and Kuhlemeyer[6] for elastic waves, at roughly the same time. In the present case it is not the entire wave that is to be absorbed, but only the longshore component of it. The usual condition for the absorption of the total wave, normally incident upon a boundary is

$$\frac{\partial \phi}{\partial n} + ik\phi = 0 \tag{23}$$

In this case the longshore component has wavenumber k_S, where $k_S = k \sin \theta$, θ being the angle of incidence of the original wave. This follows from Snell's law of refraction. The non-reflecting boundary condition is thus

$$\frac{\partial \phi}{\partial n} \pm ik \sin \theta \phi = 0 \tag{24}$$

where the minus sign is for waves meeting a boundary and the plus sign for waves leaving the boundary.

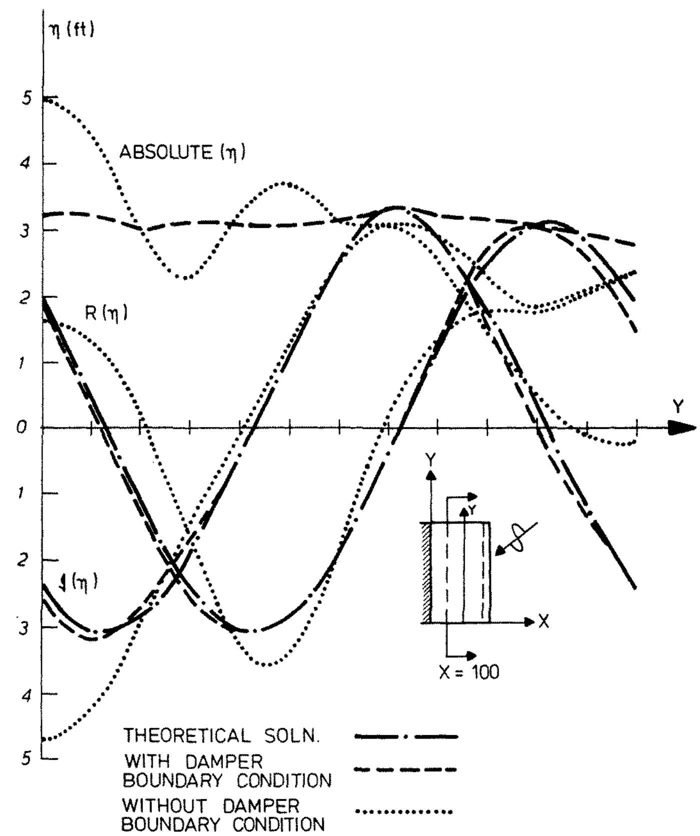

Figure 10. Wave elevation parallel to shoreline with and without damping elements on artificial boundaries. 24° angle of wave incidence

The boundary condition is equivalent to adding a term to the wave functional given by

$$\pm\frac{1}{2}\int_{\Gamma} ik \sin\theta\phi^2 \, d\Gamma \tag{25}$$

which, after finite element discretization, yields a term

$$\frac{1}{2}\int_{\Gamma_e} \mathbf{N}^T\mathbf{N}k \sin\theta \, d\Gamma \tag{26}$$

The boundary condition was applied to the same test case as used previously and was found to work well for all angles of wave incidence. Figures 9 and 10 show the wave heights across the model for deep water angles of wave incidence of 12 and 24° respectively. The theoretical plot is obtained by assuming that the central values are correct and working out

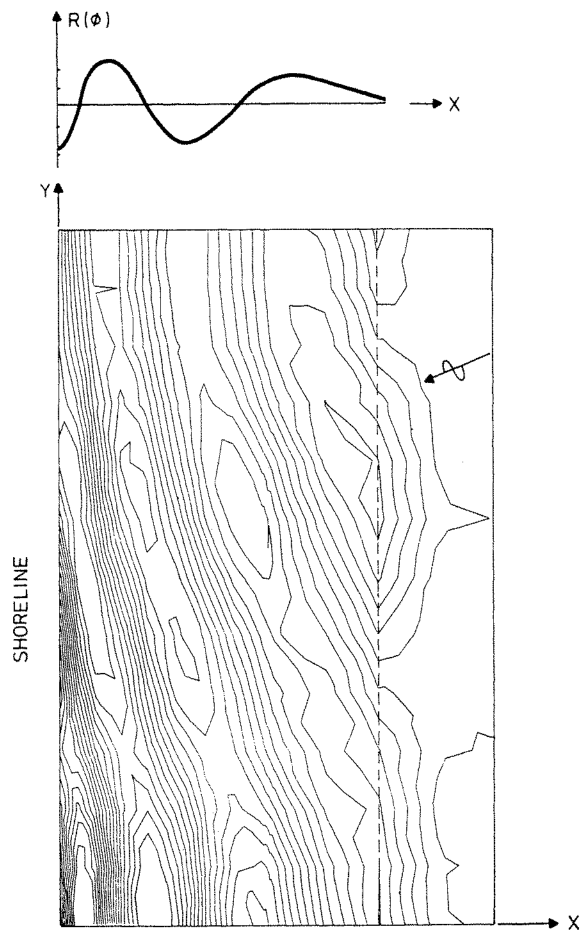

Figure 11. Contour plot of wave potentials resulting when wave program is run without damping elements on artificial boundaries. Note: wave contours are forced normal to artificial boundaries. Incident wave integral applied on dashed line

the remaining theoretical values using the phase angle at the centre and the expected phase shift across the model. The small oscillations in the absolute values are believed to be due to partial reflections off the sloping bed.

The contour plots, Figures 11 and 12, show the 24° case without and with the boundary condition being applied. The 'S' shaped contours forced by the natural boundary condition can be clearly seen. Figure 13 shows both cases plotted normal to the beach at the midpoint of the model. The plots are similar. From Figures 11 and 12 it is apparent that a normal plot at any other point would show much greater differences. However, the result does point to an alternative which is to use only the central results from a model extended considerably beyond the region of interest. More results are given by Austin.[2]

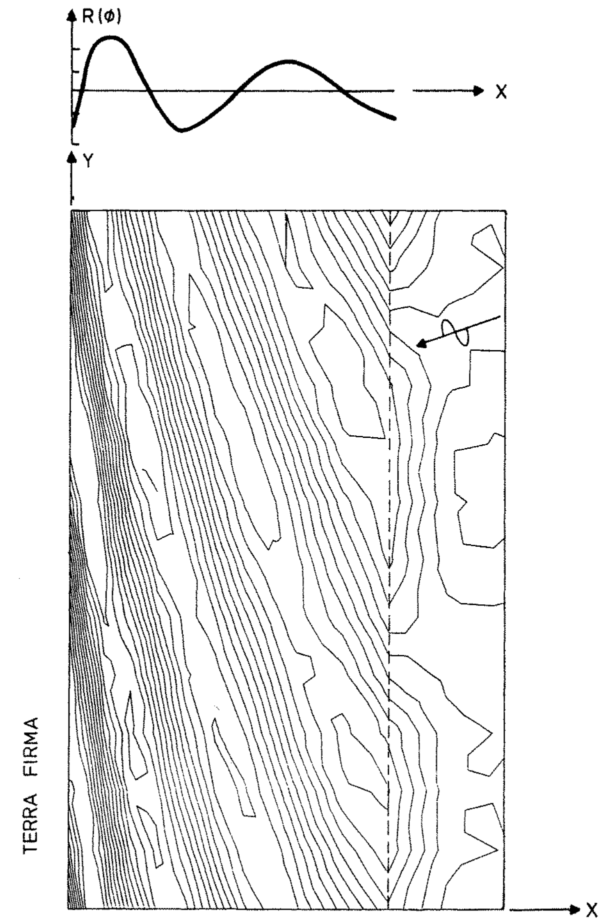

Figure 12. Contour plot of wave potentials using damping elements on artificial boundaries. 24° angle of wave incidence

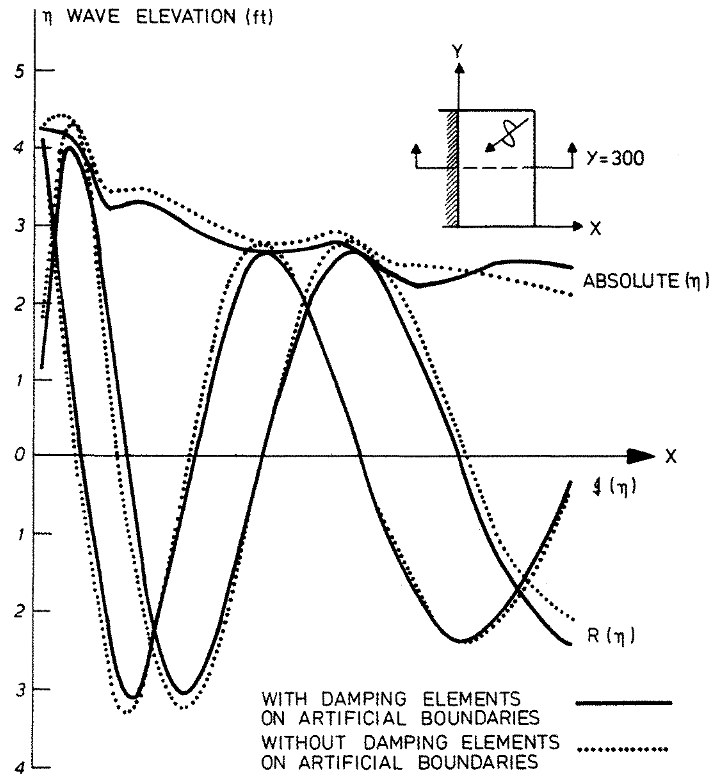

Figure 13. Wave elevation normal to shoreline with and without damping elements on artificial boundaries. 24° angle of wave incidence

CONCLUSIONS

It is highly likely that there will be increasing use of finite element models for near shore wave effects, because of their ability to model accurately wave diffraction, refraction and resonance. In such open modelling situations it is of paramount importance that the artificial finite element boundaries be modelled accurately. Three possible schemes for dealing with boundaries perpendicular to the shore have been studied in this paper. The Smith condition, despite its great success in transient problems, does not seem appropriate for periodic problems and gives bad results. The Lagrange Multiplier constraint method gives accurate results, but imposes a fixed length on the numerical model for a given angle of wave incidence. This can be extremely inconvenient, especially where the same model is to be run for a range of incident wave angles. It can be used when the bed does not slope at a constant angle. The damper method can be used for any length of model and is easy to program. Both of the successful methods have drawbacks, but the dampers seem to be the 'best buy'. Undoubtedly, the concepts used in truncating these models need further refinement, but they do enable the analyst to bound his or her model of the nearshore wave processes without adverse effects.

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